

THE SUM OF FOUR SQUARES OVER REAL QUADRATIC NUMBER FIELDS

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1. INTRODUCTION AND STATEMENT OF RESULTS

It is a well known theorem of Lagrange from 1770 that every natural number can be written as the sum of four integer squares [21]. An alternative way to phrase this is to say that the sum of four squares is a **universal** form over \mathbb{Z} ; i.e., given any $m \in \mathbb{Z}^+$ there is some integer vector \vec{x} satisfying the equation $x_1^2 + x_2^2 + x_3^2 + x_4^2 = m$. Less well known (but more powerful, as universality becomes a corollary) is the 1834 result of Jacobi providing a formula $r(m)$ for the number of ways a positive integer m can be written as a sum of four squares:

$$r(m) = 8 \sum_{0 < d|m, 4 \nmid d} d.$$

Jacobi's method of proof uses elliptic and theta functions; while distinct from later proofs involving modular forms, these objects were undeniable natural precursors.

This paper too concerns the sum of four squares; however, we replace \mathbb{Z} with O_K the ring of integers of a real quadratic number field K and \mathbb{Z}^+ with the totally positive integers O_K^+ . First is the conceptual extension of universality. Siegel [28, Theorem 1] proves that the only totally real number fields for which the sum of n integer squares is universal for some n are \mathbb{Q} and $\mathbb{Q}(\sqrt{5})$. Combining this with the results of Jacobi and Lagrange, one immediately can conclude that the only real quadratic number field over which the sum of four squares *could* be universal is $\mathbb{Q}(\sqrt{5})$. We note that while in [28] Siegel did indeed show that every $m \in \mathbb{Z}[\frac{1+\sqrt{5}}{2}]^+$ is the sum of n integer squares for some positive integer n he did not explicitly show that $n = 4$.

Continuing the historical discussion of $\mathbb{Q}(\sqrt{5})$, we note that almost twenty years before Siegel Götzký [14] provided a formula similar to Jacobi's formula over \mathbb{Z} for the number $r(m)$ of ways an integer $m \in \mathbb{Z}[\frac{1+\sqrt{5}}{2}]^+$ can be represented as the sum of four $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$ squares:

$$r(m) = 8 \sum_{(d)|(m)} N_{K/\mathbb{Q}}(d) - 4 \sum_{(2)|(d)|(m)} N_{K/\mathbb{Q}}(d) + 8 \sum_{(4)|(d)|(m)} N_{K/\mathbb{Q}}(d).$$

Again, Götzký's technique mimicked that of Jacobi in the senses that it used elliptic functions in multiple variables and can be viewed as a precursor to Hilbert modular forms. More recent results concerning the representability of $m \in \mathbb{Z}[\frac{1+\sqrt{5}}{2}]$ (or even more generally of $m \in \mathcal{O}_K^+$ for K a real quadratic number field) come from Kim and Park (see [19] and [24]); these results concern the sum of n squares to address specifically where each component of \vec{x} is distinct.

To add to this story, we begin by providing two new and short proofs first of Siegel's claim that the sum of four squares is not universal over any real quadratic number field $K \neq \mathbb{Q}(\sqrt{5})$ and second of Götzký's formula for $r(m)$ for the sum of four squares over $\mathbb{Q}(\sqrt{5})$. The latter proof in particular relies on the theory of local densities (developed by Siegel after Götzký's publication) and Hilbert modular forms.

While over number fields beyond \mathbb{Q} or $\mathbb{Q}(\sqrt{5})$, proofs of the universality of the sum of four squares are futile, one could still ask about the global representation of **locally represented** integers. That is, one could restrict attention to determining $r(m)$ for those $m \in \mathcal{O}_K^+$ which are represented in K_v for all places v . Of course, this first requires understanding which $m \in \mathcal{O}_K^+$ are locally represented by the sum of four squares.

Theorem 1. *Let K be a real quadratic number field, let Q be the sum of four squares, and let $m \in \mathcal{O}_K^+$. Then m is locally represented at all primes $\mathfrak{p} \nmid (2)$. Additionally,*

- *If $K = \mathbb{Q}(\sqrt{D})$ for $D \equiv 5 \pmod{8}$ for $D > 0$ squarefree, and if $m \in \mathcal{O}_K^+$, then m is locally represented at the even prime. Therefore, all $m \in \mathcal{O}_K^+$ are locally represented.*
- *If $K = \mathbb{Q}(\sqrt{D})$ for $D \equiv 1 \pmod{8}$ for $D > 0$ squarefree, and if $m \in \mathcal{O}_K^+$, then m is locally represented at both even primes. Therefore, all $m \in \mathcal{O}_K^+$ are locally represented.*
- *If $K = \mathbb{Q}(\sqrt{D})$ for $D \equiv 2, 3 \pmod{4}$ for $D > 0$ squarefree, and if $m \in \mathcal{O}_K^+$, then m is locally represented at the even prime \mathfrak{p}_2 if*
 - *when $D \equiv 2 \pmod{4}$ there exists an integer $k \geq 0$ so that*

$$\frac{m}{(2)^k} \equiv s \pmod{\mathfrak{p}_2^5}$$

where

$$s \in S = \{\pm 1, \pm 2, \pm 3, 4, 2\sqrt{D} \pm 2, 2\sqrt{D} \pm 1, 2\sqrt{D} \pm 3, 2\sqrt{D} + 4, 2\sqrt{D}\}.$$

- *when $D \equiv 3 \pmod{4}$ there exists an integer $k \geq 0$ so that*

$$\frac{m}{(2)^k} \equiv s \pmod{\mathfrak{p}_2^5}$$

where

$$s \in S = \{\pm 1, \pm 2\sqrt{D}, 2 \pm 2\sqrt{D}, 4\sqrt{D} \pm 1, 2\sqrt{D} \pm 1, -2\sqrt{D} \pm 1, 2, 4\sqrt{D}, 4\sqrt{D} + 2\}.$$

So now again we return to the search for Jacobi and Götzký type formulas for the number $r(m)$ of ways to express a locally represents $m \in \mathcal{O}_K^+$ as a sum of four squares. This leads us to a “known but not realized” result:

Theorem 2. *Let $K = \mathbb{Q}(\sqrt{2})$ and suppose $m \in \mathcal{O}_K^+$ is locally represented by the sum of four squares Q . The number $r_Q(m)$ of ways m is represented by Q is*

$$r_Q(m) = 8 \sum_{(d)|(m)} N_{K/\mathbb{Q}}(d) - 6 \sum_{(2)|(d)|(m)} N_{K/\mathbb{Q}}(d) + 4 \sum_{(4)|(d)|(m)} N_{K/\mathbb{Q}}(d).$$

To justify our earlier word choice, in 1960 Cohn [6] provided a formula for $r_Q(m)$ for Q the sum of four squares over $K = \mathbb{Q}(\sqrt{2})$. In his own words, Cohn mimicked the proof of Götzký (and thus our proof via local densities automatically is distinct from any previously published version). More crucially, though, his statement of the formula for $r_Q(m)$ involves cases (unlike the formula in Theorem 2) and does not mimic the results of either Götzký or Jacobi in simplicity or elegance.

Last we come to real quadratic number fields beyond $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\sqrt{2})$. It is at this point in our motivation that we must introduce modular forms; still, for purposes of introduction we merely give an overview with details and more rigorous definitions to follow. We consider the **theta series** associated to Q the sum of four squares. This is

a Hilbert modular form of level, weight and character determined by Q . The Fourier coefficients of this theta series are precisely $\{r_Q(m)\}$. Therefore understanding the values represented by Q is equivalent to understanding the Fourier coefficients of the corresponding theta series. We decompose this Hilbert modular form into its Eisenstein and cusp components which in turn allows us to express $r_Q(m) = a_E(m) + a_S(m)$ where $a_E(m)$ is the Eisenstein coefficient and $a_S(m)$ the cusp coefficient. We now come to our main result; we provide not only exact values for the Eisenstein coefficients $\{a_E(m)\}$ for any locally represented m , but also effective upper and lower bounds for these coefficients.

Theorem 3. *Let $K = \mathbb{Q}(\sqrt{D})$ for $D > 0$ a square-free integer and let $m \in \mathcal{O}_K^+$. Then there exist explicit formulas for $a_E(m)$ of the form*

$$a_E(m) = (c_{1,D}D) \cdot (c_{2,m}c_{3,D}) \cdot \left(\sum_{0 \neq (d)|(m), 2 \nmid N_{K/\mathbb{Q}}(d)} N_{K/\mathbb{Q}}(d) \right)$$

where $c_{1,D}$ and $c_{3,D}$ are constants depending only on D and where $c_{2,m}$ is a constant depending only on m . Moreover, for all locally represented m ,

$$(c_{4,D}D^{-3/2}) \cdot \left(\sum_{0 \neq (d)|(m), (2) \nmid (d)} N_{K/\mathbb{Q}}(d) \right) \leq a_E(m) \leq c_{5,m,D} \cdot \left(\sum_{0 \neq (d)|(m)} N_{K/\mathbb{Q}}(d) \right)$$

where again $c_{4,D}$ and $c_{5,m,D}$ are constants depending (respectively) upon only D and upon m and D .

One corollary to this result is the following, which provides an infinite family of quadratic number fields for which a Jacobi-style formula is not readily attainable.

Corollary 1. *For any real quadratic number field K in which 2 does not ramify, $a_E(1) < r_Q(1)$ and the theta series is not Eisenstein.*

We also have:

Corollary 2. *Let $K = \mathbb{Q}(\sqrt{D})$ for $0 < D \equiv 5 \pmod{8}$. The upper bound on $a_E(m)$ is sharp.*

This paper is organized as follows: we begin with an overview of the analytic theory of quadratic forms and the specific local density and modular forms tools we will need for the proofs of Theorems 2 and 3; this of course will include a proof of Theorem 1. We proceed then with the short proof of the non-universality of the sum of four squares over real quadratic number fields $K \neq \mathbb{Q}(\sqrt{5})$, and we derive Götzky's formula for $r_Q(m)$ using local density and Hilbert modular form theory. We then prove Theorem 1 and Theorem 3. After treating all general cases, we will return to the proof of Theorem 2. Last, to give a complete and thorough set of examples related to Theorem 3 we end with a section of additional applications in which we provide the theta series decomposition of the sum of four squares over $\mathbb{Q}(\sqrt{3})$, $\mathbb{Q}(\sqrt{13})$ and $\mathbb{Q}(\sqrt{17})$.

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1.1. Basic Definitions. For details and additional background we refer the reader to [16] and [30].

Let K be a number field (allowing $K = \mathbb{Q}$) with \mathcal{O}_K its ring of integers. An n -ary quadratic form over \mathcal{O}_K is a form

$$Q(\vec{x}) = Q(x_1, \dots, x_n) = \sum_{1 \leq i \leq j \leq n} a_{ij}x_i x_j \in \mathcal{O}_K[x_1, \dots, x_n].$$

By M_Q we denote the symmetric $n \times n$ matrix with coefficients in $\frac{1}{2}\mathcal{O}_K$ representing Q , in which case we note that $Q(\vec{x}) = \vec{x}^T M_Q \vec{x}$.

Now $Q : \mathcal{O}_K^n \rightarrow \mathcal{O}_K$ be a quadratic form over a totally real number field K of degree $[K : \mathbb{Q}] = d$. Let $\sigma_1, \dots, \sigma_d$ denote the distinct embeddings from K to \mathbb{R} . We say that Q is **(totally positive definite)** if the following conditions hold:

- (1) $Q(\vec{x}) = 0$ if and only if $\vec{x} = 0$.
- (2) For all $\vec{x} \neq 0$, $\sigma_i(Q(\vec{x})) > 0$ for all $1 \leq i \leq d$.

We denote by O_K^+ the totally positive elements of O_K ; i.e., $O_K^+ = \{m \in O_K \mid \sigma_i(m) > 0 \forall \sigma_i : K \hookrightarrow \mathbb{R}\}$. Given a positive definite n -ary quadratic form Q over such a number field K , if for all $m \in O_K^+$ there exists a solution $Q(\vec{x}) = m$ with $\vec{x} \in O_K^n$ we say that Q is **universal**.

For the rest of this paper, unless otherwise specified, we consider K to be a real quadratic number field, and Q to be a totally positive definite quaternary quadratic form over K . For notational purposes, we will be assuming that K has narrow class number one; for a full a detailed treatment in the most general setting, we specifically refer readers to [30, Section 7].

We define the **norm** N_Q of Q to be the largest O_K ideal such that for all primes \mathfrak{p} , $N_{\mathfrak{p}}(2M_{Q,\mathfrak{p}})^{-1}$ is a matrix of integral ideals with diagonal entries in $2O_K$. We also define the **determinant** \mathcal{D}_Q of Q to be $\det(M_Q)$. Last, we set the following Hecke character defined for all $\mathfrak{p} \nmid 2N_Q$ be

$$\chi_Q(\mathfrak{p}) = \left(\frac{\mathcal{D}_Q}{\mathfrak{p}} \right).$$

Then define the **theta series** associated to Q as

$$\Theta_Q(z) = 1 + \sum_{m \in (\mathfrak{h}^{-1})^+} r_Q(m) e^{2\pi i \text{Tr}(m \cdot z)},$$

where the sum runs over all totally positive elements of the inverse different ideal.

Theorem: $\Theta_Q(m)$ is a Hilbert modular form of (parallel) weight 2 over $\Gamma_0(N_Q)$ with associated character χ_Q .

Proof. See [1, Theorem 2.2, pg. 61]. □

As each space of modular forms of fixed weight, level and character decomposes into a direct sum of Eisenstein series and cusp forms, we write $\Theta_Q(m) = E_Q(z) + S_Q(z)$ where $E_Q(z)$ is Eisenstein and $S_Q(z)$ is a cusp form. We then consider

$$r_Q(m) = a_E(m) + a_S(m)$$

for each coefficient in the Fourier expansion of Θ_Q .

N.B. For Q the sum of four squares the character χ_Q will always be trivial. Moreover, the level ideal N_Q is $(4)O_K$. Thus $\Theta_Q(z)$ is a Hilbert modular form of parallel weight two, level (4) and trivial character over K .

We now recall some terminology from Hanke [16] relevant specifically to later local density computations.

Let $R_{\mathfrak{p}^v}(m) := \{\vec{x} \in (O_K/\mathfrak{p}^v O_K)^4 : Q(\vec{x}) \equiv m \pmod{\mathfrak{p}^v}\}$ and set $\#R_{\mathfrak{p}^v}(m) = r_{\mathfrak{p}^v}(m)$. We say $\vec{x} \in R_{\mathfrak{p}^v}(m)$ is of **Zero type** if $\vec{x} \equiv \vec{0} \pmod{\mathfrak{p}}$ (in which case, we say $\vec{x} \in R_{\mathfrak{p}^v}^{\text{Zero}}(m)$ with $\#R_{\mathfrak{p}^v}^{\text{Zero}}(m) := r_{\mathfrak{p}^v}^{\text{Zero}}(m)$), is of **Good type** if $\mathfrak{p}^{v_j} \vec{x}_j \not\equiv \vec{0} \pmod{\mathfrak{p}}$ for some j (in which case, we say $\vec{x} \in R_{\mathfrak{p}^v}^{\text{Good}}(m)$ with $\#R_{\mathfrak{p}^v}^{\text{Good}}(m) := r_{\mathfrak{p}^v}^{\text{Good}}(m)$), and is of **Bad type** otherwise.

For Q the sum of four squares, we will only need to consider Zero and Good type solutions. We will need, however, the following reduction maps:

Theorem

$$r_{\mathfrak{p}^{k+\ell}}^{\text{Good}}(m) = N(\mathfrak{p})^{3\ell} r_{\mathfrak{p}^k}^{\text{Good}}(m)$$

for $k \geq 2\text{ord}_{\mathfrak{p}}(2) + 1$.

Proof. See [16, Lemma 3.2]. □

Theorem The map

$$\begin{aligned}\pi_Z : R_{\mathfrak{p}^k}^{\text{Zero}}(m) &\rightarrow R_{\mathfrak{p}^{k-2}}\left(\frac{m}{\pi_{\mathfrak{p}}^2}\right) \\ \vec{x} &\mapsto \pi_{\mathfrak{p}}^{-1} \vec{x} \pmod{\mathfrak{p}^{k-2}}\end{aligned}$$

is a surjective map with multiplicity $N_{K/\mathbb{Q}}(\mathfrak{p})^4$.

Proof. See [16, pg. 359]. □

2. COMPUTING LOCAL DENSITIES AND PROOF OF THEOREM 1

In this section, we introduce Siegel's theory of local densities which we use in our later proofs of Theorems 3 and 2.

Theorem (Siegel): Let Q be a positive definite quadratic form over O_K . Let $r_Q(m) = a_E(m) + a_S(m)$ be the decomposition of the Fourier coefficients of the theta series associated to Q into Eisenstein and cusp components.

$$a_E(m) = \prod_{\mathfrak{p}} \beta_{\mathfrak{p}}(m)$$

where \mathfrak{p} runs over all places of K and where $\beta_{\mathfrak{p}}(m)$ is the \mathfrak{p} -adic local density, defined to be

$$\beta_{\mathfrak{p}}(m) := \lim_{U \rightarrow \{m\}} \frac{\text{Vol}(Q^{-1}(U))}{\text{Vol}(U)}$$

where U is an open neighborhood of m in $K_{\mathfrak{p}}$ and where the volume is determined by a fixed Haar measure.

Corollary: When \mathfrak{p} is a finite place and Q is a quaternary form, we have

$$\beta_{\mathfrak{p}}(m) = \lim_{v \rightarrow \infty} \frac{r_{\mathfrak{p}^v}(m)}{N_{K/\mathbb{Q}}(\mathfrak{p})^{3v}}.$$

Using this, we write

$$\begin{aligned}a_E(m) &= \prod_{\mathfrak{p}} \beta_{\mathfrak{p}}(m) \\ &= \underbrace{\left(\prod_{\mathfrak{p}|\infty} \beta_{\mathfrak{p}}(m) \right)}_{(I)} \underbrace{\left(\prod_{\mathfrak{p} \nmid N_Q m} \beta_{\mathfrak{p}}(m) \right)}_{(II)} \underbrace{\left(\prod_{\mathfrak{p} | N_Q} \beta_{\mathfrak{p}}(m) \right)}_{(III)} \underbrace{\left(\prod_{\mathfrak{p} | m, \mathfrak{p} \nmid N_Q} \beta_{\mathfrak{p}}(m) \right)}_{(IV)}.\end{aligned}$$

We now discuss each of the four terms in the above product.

2.0.1. *The infinite primes (I).* We refer to another theorem of Siegel:

Theorem [27, Hilfssatz 72] Let K be a totally real number field with discriminant Δ with $h = [K : \mathbb{Q}]$. Let $m \in O_K^+$ and let σ be the discriminant of a quaternary positive definite quadratic form Q defined over O_K . Then

$$\prod_{\mathfrak{p}|\infty} \beta_{\mathfrak{p}}(m) = \pi^{2h} \Delta^{-3/2} (N_{K/\mathbb{Q}}(\sigma))^{-1/2} N_{K/\mathbb{Q}}(m).$$

In particular, when Q is the sum of four squares over a real quadratic number field, this formula yields

$$\prod_{\mathfrak{p}|\infty} \beta_{\mathfrak{p}}(m) = \pi^4 \Delta^{-3/2} N_{K/\mathbb{Q}}(m)$$

where $\Delta = D$, for $K = \mathbb{Q}(\sqrt{D})$, $D \equiv 1 \pmod{4}$, and $\Delta = 4D$ otherwise.

2.0.2. *The infinite product of primes (II).* Primary resources for the simplification of this term are [16], [18] and [20]. The main result is:

Theorem: Let Q be a quaternary positive definite quadratic form with level ideal N_Q and with character χ_Q defined over a totally real number field K . Let $m \in \mathcal{O}_K^+$. Then

$$\prod_{\mathfrak{p} \nmid N_Q m} \beta_{\mathfrak{p}}(m) = L_K(2, \chi_Q)^{-1} \left(\prod_{\mathfrak{p} \mid N_Q m} \frac{N_{K/\mathbb{Q}}(\mathfrak{p})^2}{N_{K/\mathbb{Q}}(\mathfrak{p})^2 - \chi_Q(\mathfrak{p})} \right).$$

Proof. While not explicitly stated as a theorem, details on this result can be found in [16, pg. 362-363]. \square

Now, however, an exact value for $L_K(2, \chi_Q)^{-1}$ must be obtained. As for Q the sum of four squares, this L -function is simply the ζ function, we restrict our attention to that case.

Theorem: Let K be a quadratic number field with discriminant Δ . Then

$$\zeta_K(2) = \zeta_{\mathbb{Q}}(2) L_{\mathbb{Q}}(2, \chi_{\Delta})$$

where χ_{Δ} is the quadratic Dirichlet character of conductor $|\Delta|$.

Proof. The proof of this statement is fairly direct, and is in fact given as an exercise in [4]. \square

Methods for computing special values of L -functions over \mathbb{Q} are found in Iwasawa [18]. In particular, we have

Theorem: Let K be a real quadratic number field of discriminant Δ . Let χ_{Δ} be the quadratic Dirichlet character of conductor Δ . Then

$$L_{\mathbb{Q}}(2, \chi_{\Delta}) = -\frac{2\pi^2 \tau(\chi_{\Delta})}{\Delta^2} L_{\mathbb{Q}}(1-2, \chi_{\Delta}),$$

where $\tau(\chi_{\Delta})$ denotes the traditional Gauss sum $\tau(\chi_{\Delta}) = \sum_{a=1}^{\Delta} \chi_{\Delta}(a) e^{2\pi i a/\Delta}$ and where

$$L_{\mathbb{Q}}(1-2, \chi_{\Delta}) = -\frac{1}{2\Delta} \sum_{a=1}^{\Delta} \chi_{\Delta}(a) a(a-\Delta).$$

Proof. This is a special case of a combination of [18, pg. 5 and pg. 11 Theorem 1]. \square

In practice, then, we have

$$\prod_{\mathfrak{p} \nmid N_Q m} \beta_{\mathfrak{p}}(m) = \frac{6\Delta^{5/2}}{\pi^4} \left(\sum_{a=1}^{\Delta} \chi_{\Delta}(a) a(a-\Delta) \right)^{-1} \left(\prod_{\mathfrak{p} \mid N_Q m} \frac{N_{K/\mathbb{Q}}(\mathfrak{p})^2}{N_{K/\mathbb{Q}}(\mathfrak{p})^2 - 1} \right).$$

2.0.3. *Primes dividing the level (III).* Again, we concentrate solely on Q the sum of four squares over real quadratic number fields K . As stated earlier, $N_Q = (4)\mathcal{O}_K$ so the only primes to consider are the even primes.

Lemma 1. Let $K = \mathbb{Q}(\sqrt{D})$, $0 < D \equiv 5 \pmod{8}$ with D squarefree. Suppose m is locally represented by the sum of four squares Q . Then

$$\beta_{(2)}(m) = \begin{cases} 1, & \text{ord}_{(2)}(m) = 0 \\ \frac{15}{8} \left(\sum_{i=0}^{N-1} \frac{1}{4^{2i}} \right) + \frac{3}{4^{2N+1}}, & \text{ord}_{(2)}(m) = 2N+1, N \geq 0 \\ \frac{15}{8} \left(\sum_{i=0}^{N-2} \frac{1}{4^{2i}} \right) + \frac{27}{4^{2N}}, & \text{ord}_{(2)}(m) = 2N, N > 0. \end{cases}$$

Proof. When $(2) \nmid (m)$, all solutions are of Good type and we have:

$$\begin{aligned}\beta_{(2)}(m) &= \lim_{v \rightarrow \infty} \frac{r_{(2)^v}^{\text{Good}}(m)}{4^{3v}} \\ &= \lim_{v \rightarrow \infty} \frac{4^{3(v-3)} r_{(8)}^{\text{Good}}(m)}{4^{3v}} \\ &= \frac{262144}{2^{18}} = 1.\end{aligned}$$

For $\text{ord}_2(m) = 2N + 1$, $N \geq 0$, we have Good and Zero type solutions with

$$\begin{aligned}\beta_{(2)}(m) &= \lim_{v \rightarrow \infty} \frac{r_{(2)^v}^{\text{Good}}(m)}{4^{3v}} + \lim_{v \rightarrow \infty} \frac{r_{(2)^v}^{\text{Zero}}(m)}{4^{3v}} \\ &= \lim_{v \rightarrow \infty} \frac{4^{3(v-3)} r_{(8)}^{\text{Good}}(m)}{4^{3v}} + \lim_{v \rightarrow \infty} \frac{1}{4^{3v}} \left(\sum_{i=1}^N 4^{4i} r_{(2)^{v-2i}}^{\text{Good}} \left(\frac{m}{2^{2i}} \right) \right) \\ &= \frac{15}{8} + \frac{\sum_{i=1}^{N-1} 2^{15} \cdot 3 \cdot 5}{2^{18} \cdot 4^{2i}} + \frac{2^{16} \cdot 3}{4^9 \cdot 4^{2N}} = \frac{15}{8} \left(\sum_{i=0}^{N-1} \frac{1}{4^{2i}} \right) + \frac{3}{4^{2N+1}}.\end{aligned}$$

Last, when $\text{ord}_2(m) = 2N$, $N \geq 1$, we have Good and Zero type solutions, and

$$\begin{aligned}\beta_{(2)}(m) &= \lim_{v \rightarrow \infty} \frac{r_{(2)^v}^{\text{Good}}(m)}{4^{3v}} + \lim_{v \rightarrow \infty} \frac{r_{(2)^v}^{\text{Zero}}(m)}{4^{3v}} \\ &= \frac{15}{8} + \frac{1}{4^9} \left(\sum_{i=1}^N 4^{-2i} r_{(8)}^{\text{Good}} \left(\frac{m}{2^{2i}} \right) \right) \\ &= \frac{15}{8} + \sum_{i=1}^{N-2} \frac{r_{(8)}^{\text{Good}} \left(\frac{m}{2^{2i}} \right)}{4^{2i+9}} + \frac{r_{(8)}^{\text{Good}} \left(\frac{m}{2^{2(N-1)}} \right)}{4^{9+2(N-1)}} + \frac{r_{(8)}^{\text{Good}} \left(\frac{m}{2^{2N}} \right)}{4^{2N+9}} \\ &= \frac{15}{8} + \frac{15}{8} \left(\sum_{i=1}^{N-2} 4^{-2i} \right) + \frac{2^{15} \cdot 13}{2^{18} \cdot 4^{2(N-1)}} + \frac{2^{18}}{2^{18} \cdot 4^{2N}} = \frac{15}{8} \left(\sum_{i=0}^{N-2} \frac{1}{4^{2i}} \right) + \frac{27}{4^{2N}}.\end{aligned}$$

□

Lemma 2. Let $K = \mathbb{Q}(\sqrt{D})$, $0 < D \equiv 1 \pmod{8}$ with D squarefree. Suppose m is locally represented by the sum of four squares Q . Let $(2) = \mathfrak{p}_1 \mathfrak{p}_2$. Then for $i = 1, 2$

$$\beta_{\mathfrak{p}_i}(m) = \begin{cases} 1, & \text{ord}_{\mathfrak{p}_i}(m) = 0 \\ \frac{3}{2^{2N+1}}, & \text{ord}_{\mathfrak{p}_i}(m) = 2N + 1, N \geq 0 \\ \frac{3}{2^{2N}}, & \text{ord}_{\mathfrak{p}_i}(m) = 2N, N \geq 1. \end{cases}$$

Proof. When $\text{ord}_{\mathfrak{p}_i}(m) = 0$, then all solutions are of Good type and

$$\begin{aligned}\beta_{\mathfrak{p}_i}(m) &= \lim_{v \rightarrow \infty} \frac{r_{\mathfrak{p}_i^v}^{\text{Good}}(m)}{2^{3v}} \\ &= \frac{2^{3(v-3)} r_{\mathfrak{p}_i^3}^{\text{Good}}(m)}{2^{3v}} = \frac{2^9}{2^9} = 1.\end{aligned}$$

For $\text{ord}_{p_i}(m) = 2N + 1, N \geq 0$, we have Good and Zero type solutions with

$$\begin{aligned}
\beta_{p_i}(m) &= \lim_{v \rightarrow \infty} \frac{r_{p_i^v}^{\text{Good}}(m)}{2^{3v}} + \lim_{v \rightarrow \infty} \frac{r_{p_i^v}^{\text{Zero}}(m)}{2^{3v}} \\
&= 0 + \lim_{v \rightarrow \infty} \frac{1}{2^{3v}} \left(\sum_{i=1}^N 2^{4i} r_{p_i^{v-2i}}^{\text{Good}}(m/p_i^2) \right) \\
&= 2^{4N} \lim_{v \rightarrow \infty} \frac{2^{3(v-2N-3)} r_{p_i^3}^{\text{Good}}(m/p_i^{2N})}{2^{3v}} \\
&= \frac{2^8 \cdot 3}{2^9 \cdot 2^{2N}} = \frac{3}{2^{2N+1}}.
\end{aligned}$$

Last, suppose $\text{ord}_{p_i}(m) = 2N, N \geq 1$. We have Good and Zero type solutions with

$$\begin{aligned}
\beta_{p_i}(m) &= \lim_{v \rightarrow \infty} \frac{r_{p_i^v}^{\text{Good}}(m)}{2^{3v}} + \lim_{v \rightarrow \infty} \frac{r_{p_i^v}^{\text{Zero}}(m)}{2^{3v}} \\
&= 0 + \lim_{v \rightarrow \infty} \frac{1}{2^{3v}} \left(\sum_{i=1}^N 2^{4i} r_{p_i^{v-2i}}^{\text{Good}}(m/p_i^2) \right) \\
&= \lim_{v \rightarrow \infty} \frac{2^{4(N-1)} 2^{3(v-2(N-1)-3)} r_{p_i^3}^{\text{Good}}(m/2^{2(N-1)})}{2^{3v}} + \lim_{v \rightarrow \infty} \frac{2^{4N} 2^{3(v-2N-3)} r_{p_i^3}^{\text{Good}}(m/2^{2N})}{2^{3v}} \\
&= \frac{r_{p_i^3}^{\text{Good}}(m/2^{2(N-1)})}{2^9 2^{2(N-1)}} + \frac{r_{p_i^3}^{\text{Good}}(m/2^{2N})}{2^9 2^{2N}} \\
&= \frac{2^8}{2^9 2^{2(N-1)}} + \frac{2^9}{2^9 2^{2N}} = \frac{3}{2^{2N}}.
\end{aligned}$$

□

Lemma 3. Let $K = \mathbb{Q}(\sqrt{D}), 0 < D \equiv 2, 3 \pmod{4}$ with D squarefree. Last suppose m is locally represented by the sum of four squares Q . Then $(2) = p_2^2$ and

$$\beta_{p_2}(m) = \begin{cases} 2, & \text{ord}_{p_2}(m) = 0, 2 \\ \frac{9}{4} \left(\frac{2^{2(N-1)} - 1}{2^{2(N-1)} - 2^{2(N-2)}} \right) + \frac{3}{2^{2(N-1)+1}}, & \text{ord}_{p_2}(m) = 2N + 1, N \geq 1 \\ \frac{9}{4} \left(\frac{2^{2(N-2)} - 1}{2^{2(N-2)} - 2^{2(N-3)}} \right) + \frac{1}{2^{2(N-2)}}, & \text{ord}_{p_2}(m) = 2N, N \geq 2. \end{cases}$$

Proof. Using the notation of Hanke [16], when $\text{ord}_{p_2}(m) = 0, 2$, all solutions will be of Good type and

$$\beta_{p_2}(m) = \lim_{v \rightarrow \infty} \frac{r_{p_2^v}^{\text{Good}}(m)}{2^{3v}} = \frac{2^{16}}{2^{15}} = 2.$$

Next suppose $\text{ord}_{p_2}(m) = 2N + 1, N \geq 1$ (Note $N = 0$ is not possible, as such m are not locally represented). Then there is the potential for both Zero type and Good type solutions, and

$$\begin{aligned}
\beta_{p_2}(m) &= \lim_{v \rightarrow \infty} \frac{r_{p_2^v}^{\text{Good}}(m)}{2^{3v}} + \lim_{v \rightarrow \infty} \frac{r_{p_2^v}^{\text{Zero}}(m)}{2^{3v}} \\
&= \frac{2^{13} \cdot 9}{2^{15}} + \lim_{v \rightarrow \infty} \left(\sum_{i=1}^{N-2} 2^{4i} 2^{3(v-2i-5)} r_{p_2^5}^{\text{Good}}(m/p_2^{2i}) \right) + \lim_{v \rightarrow \infty} \frac{1}{2^{3v}} (2^{4(N-1)} 2^{3(v-2(N-1)-5)} r_{p_2^5}^{\text{Good}}(m/p_2^{2N-2})) \\
&= \frac{9}{4} \left(\sum_{i=0}^{N-2} \frac{1}{2^{2i}} \right) + \frac{3}{2^{2(N-1)+1}}
\end{aligned}$$

and the result follows. Last suppose $\text{ord}_{p_2}(m) = 2N$, $N \geq 2$. Then both Good and Zero type solutions exist with

$$\begin{aligned}
\beta_{p_2}(m) &= \lim_{v \rightarrow \infty} \frac{r_{p_2^v}^{\text{Good}}(m)}{2^{3v}} + \lim_{v \rightarrow \infty} \frac{r_{p_2^v}^{\text{Zero}}(m)}{2^{3v}} \\
&= \frac{2^{13}9}{2^{15}} + \lim_{v \rightarrow \infty} \frac{1}{2^{3v}} \left(\sum_{i=1}^{N-3} 2^{4i} 2^{3(v-2i-5)} r_{p_2^5}^{\text{Good}}(m/p_2^{2i}) \right) + \lim_{v \rightarrow \infty} \frac{1}{2^{3v}} \left(2^{4(N-2)} 2^{3(v-2(N-2)-5)} r_{p_2^5}^{\text{Good}}(m/p_2^{2(N-2)}) \right) \\
&\quad + \lim_{v \rightarrow \infty} \frac{1}{2^{3v}} \left(2^{4(N-1)} 2^{3(v-2(N-1)-5)} r_{p_2^5}^{\text{Good}}(m/p_2^{2(N-1)}) \right) \\
&= \frac{9}{4} \left(\sum_{i=0}^{N-3} \frac{1}{2^{2i}} \right) + \frac{7}{4} \cdot \frac{1}{2^{2(N-2)}} + \frac{1}{2^{2N-3}}
\end{aligned}$$

and the claim holds. \square

2.0.4. *The remaining primes (IV).* Again, we restrict our attention to the sum of four squares Q over a real number field K , noting that $\chi_Q(\mathfrak{p}) \equiv 1$.

Lemma 4. *Let $m \in \mathcal{O}_K^+$ be locally represented by the sum of four squares Q . For $\mathfrak{p} \nmid (2)$, $\mathfrak{p} \nmid m$,*

$$\frac{\beta_{\mathfrak{p}}(m) N_{K/\mathbb{Q}}(\mathfrak{p})^2}{N_{K/\mathbb{Q}}(\mathfrak{p})^2 - \chi_Q(\mathfrak{p})} = \sum_{i=0}^{\text{ord}_{\mathfrak{p}}(m)} N_{K/\mathbb{Q}}(\mathfrak{p})^{-i}.$$

Proof. Suppose $\text{ord}_{\mathfrak{p}}(m) = 2N$, $N \in \mathbb{N}$. Here both Good and Zero type solutions exist and

$$\begin{aligned}
\beta_{\mathfrak{p}}(m) &= \lim_{v \rightarrow \infty} \frac{r_{\mathfrak{p}^v}^{\text{Good}}(m)}{N_{K/\mathbb{Q}}(\mathfrak{p})^{3v}} + \lim_{v \rightarrow \infty} \frac{r_{\mathfrak{p}^v}^{\text{Zero}}(m)}{N_{K/\mathbb{Q}}(\mathfrak{p})^{3v}} \\
&= \frac{N_{K/\mathbb{Q}}(\mathfrak{p})^3 + N_{K/\mathbb{Q}}(\mathfrak{p})(N_{K/\mathbb{Q}}(\mathfrak{p}) - 1) - 1}{N_{K/\mathbb{Q}}(\mathfrak{p})^3} + \lim_{v \rightarrow \infty} \frac{1}{N_{K/\mathbb{Q}}(\mathfrak{p})^{3v}} \left(\left(\sum_{i=1}^{N-1} N_{K/\mathbb{Q}}(\mathfrak{p})^{4i} r_{\mathfrak{p}^{v-2i}}^{\text{Good}}(m/p^{2i}) \right) + N_{K/\mathbb{Q}}(\mathfrak{p})^{4N} r_{\mathfrak{p}^{v-2N}}^{\text{Good}}(m/p^{2N}) \right) \\
&= \left(\sum_{i=0}^{N-1} N_{K/\mathbb{Q}}(\mathfrak{p})^{-2i} \right) \left(\frac{N_{K/\mathbb{Q}}(\mathfrak{p})^3 + N_{K/\mathbb{Q}}(\mathfrak{p})(N_{K/\mathbb{Q}}(\mathfrak{p}) - 1) - 1}{N_{K/\mathbb{Q}}(\mathfrak{p})^3} \right) + N_{K/\mathbb{Q}}(\mathfrak{p})^{-2N} \left(1 - \frac{1}{N_{K/\mathbb{Q}}(\mathfrak{p})^2} \right).
\end{aligned}$$

So for such primes

$$\begin{aligned}
\frac{\beta_{\mathfrak{p}}(m) N_{K/\mathbb{Q}}(\mathfrak{p})^2}{N_{K/\mathbb{Q}}(\mathfrak{p})^2 - \chi_Q(\mathfrak{p})} &= \left(\sum_{i=0}^{N-1} N_{K/\mathbb{Q}}(\mathfrak{p})^{-2i} \right) \left(1 + \frac{1}{N_{K/\mathbb{Q}}(\mathfrak{p})} \right) + N_{K/\mathbb{Q}}(\mathfrak{p})^{-2N} \\
&= \sum_{i=0}^{2N} N_{K/\mathbb{Q}}(\mathfrak{p})^{-i}.
\end{aligned}$$

The proof for $\text{ord}_{\mathfrak{p}}(m)$ odd behaves identically. \square

Naturally, we still have not addressed the proof of which $m \in \mathcal{O}_K^+$ are locally represented. We refer the reader to the statement of Theorem 1 and sketch its proof.

Proof. (of Theorem 1) Given $m \in \mathcal{O}_K^+$ and \mathfrak{p} prime, by [16, Lemma 3.4] to check if m is locally represented at \mathfrak{p} , it suffices to check if quotients of m by square factors are represented $(\text{mod } \mathfrak{p})$ when $\mathfrak{p} \nmid N$ and $(\text{mod } \mathfrak{p}^{\text{ord}_{\mathfrak{p}}(4N_Q)+2})$ when $\mathfrak{p} \mid N_Q$.

The remainder of the proof can be completed via computer software package such as Sage [26]. We outline a particular instance of an algorithm which will quickly compute the local density. Consider the case of $D \equiv 1 \pmod{8}$. Here (2) splits as $(2) = \mathfrak{p}_1 \mathfrak{p}_2$. Consider also a case where $\text{ord}_{\mathfrak{p}_i}(m) = 0$. All solutions will be of Good type, and so one only need compute $r_{\mathfrak{p}_i^3}(m)$:

Step 1. Compute $R_{\mathfrak{p}_i}(1)$, noting that $(\mathcal{O}_K/\mathfrak{p}_i \mathcal{O}_K) \cong \mathbb{F}_2$.

Step 2. Consider the perfect squares $(\text{mod } p_i^3)$ and their reductions $(\text{mod } p_i)$.

Step 3. For each $\vec{x} \in R_{p_i}(1)$ consider all lifts to $\vec{y} \in R_{p_i^3}(m)$.

Again, the other cases will behave similarly. □

3. A NON-UNIVERSALITY PROOF

We recall the statement of Siegel regarding non-universality of the sum of four squares:

Theorem: [28] Let K be a totally real number field and suppose that all totally positive integers are sums of integral squares in K ; then K is either the rational number field or the real quadratic number field $\mathbb{Q}(\sqrt{5})$.

We provide a proof specifically for the sum of four squares.

Proof. Let $D > 0$ be a square free integer, and let $K = \mathbb{Q}(\sqrt{D})$. If $D \equiv 2, 3 \pmod{4}$, then $\mathcal{O}_K = \mathbb{Z}[\sqrt{D}]$. Consequently, if $a + b\sqrt{D} = m \in \mathcal{O}_K^+$ is a sum of squares, b is necessarily even. Thus the totally positive integer $m = \lceil \sqrt{D} \rceil + \sqrt{D}$ cannot be expressed as a sum of four \mathcal{O}_K squares.

Last if $D \equiv 1 \pmod{4}$, $D > 5$ and $\mathcal{O}_K = \mathbb{Z}[\frac{1+\sqrt{D}}{2}]$ then again using field arithmetic one can show that $m = \lfloor \frac{1+\sqrt{D}}{2} \rfloor + \frac{1+\sqrt{D}}{2}$ cannot be expressed as a sum of four \mathcal{O}_K squares. □

4. A LOCAL DENSITIES PROOF OF GÖTZKY'S FORMULA

We begin with a formal statement of the result in question:

Theorem: Let $K = \mathbb{Q}(\sqrt{5})$. The number $r_Q(m)$ of ways a totally positive integer m can be expressed as the sum of four \mathcal{O}_K -squares is

$$r_Q(m) = 8 \sum_{(d)|(m)} N_{K/\mathbb{Q}}(d) - 4 \sum_{(2)|(d)|(m)} N_{K/\mathbb{Q}}(d) + 8 \sum_{(4)|(d)|(m)} N_{K/\mathbb{Q}}(d).$$

Proof. For the sake of brevity, we provide merely a sketch; complete details can be found in [29, Chapter 4].

By Theorem 1 all $m \in \mathcal{O}_K^+$ are locally represented. Next, using the lemmas of the previous section, we claim

$$\begin{aligned} a_E(m) &= 8N_{K/\mathbb{Q}}(m)\beta_{(2)}(m) \left(\prod_{p|m, p \nmid (2)} \frac{N_{K/\mathbb{Q}}(p)^2 \beta_p(m)}{N_{K/\mathbb{Q}}(p)^2 - 1} \right) \\ &= \begin{cases} 8 \cdot \left(\sum_{(0) \neq (d)|(m)} N_{K/\mathbb{Q}}(d) \right), & \text{ord}_{(2)}(m) = 0 \\ 8 \cdot 3 \cdot \left(1 + 10 \sum_{i=0}^{N-1} 4^{2(N-i+1)+1} \right) \cdot \left(\sum_{(0) \neq (d)|(m), (2) \nmid (d)} N_{K/\mathbb{Q}}(d) \right), & \text{ord}_{(2)}(m) = 2N+1, N \geq 0 \\ 8 \cdot 3 \cdot \left(9 + 10 \sum_{i=0}^{N-2} 4^{2(N-i+1)} \right) \cdot \left(\sum_{(0) \neq (d)|(m), (2) \nmid (d)} N_{K/\mathbb{Q}}(d) \right), & \text{ord}_{(2)}(m) = 2N, N \geq 1. \end{cases} \end{aligned}$$

The space of Hilbert modular cusp forms over $\mathbb{Q}(\sqrt{5})$ of parallel weight 2, level (4) and trivial character is zero dimensional [22]. Thus, $r_Q(m) = a_E(m)$ for all $m \in \mathcal{O}_K^+$. To show the formula for $a_E(m)$ above can be rearranged to give the formula for $r_Q(m)$ provided in statement of the theorem, we proceed by cases.

When $\text{ord}_{(2)}(m) = 0$, then certainly

$$8 \cdot \left(\sum_{(d)|(m)} N_{K/\mathbb{Q}}(d) \right) = 8 \sum_{(d)|(m)} N_{K/\mathbb{Q}}(d) - 4 \sum_{2|(d)|(m)} N_{K/\mathbb{Q}}(d) + 8 \sum_{4|(d)|(m)} N_{K/\mathbb{Q}}(d).$$

For the remaining two cases, we write

$$8 \sum_{(d)|(m)} N_{K/\mathbb{Q}}(d) - 4 \sum_{2|(d)|(m)} N_{K/\mathbb{Q}}(d) + 8 \sum_{4|(d)|(m)} N_{K/\mathbb{Q}}(d) = 8 \sum_{(d)|(m), \text{ord}_{(2)}(d)=0} N_{K/\mathbb{Q}}(d) + 4 \sum_{(d)|(m), \text{ord}_{(2)}(d)=1} N_{K/\mathbb{Q}}(d) + 12 \sum_{(d)|(m), \text{ord}_{(2)}(d) \geq 2} N_{K/\mathbb{Q}}(d).$$

We now note

$$8 \cdot 3 \cdot \left(1 + 10 \sum_{i=0}^{N-1} 4^{2(N-i)+1} \right) \cdot \left(\sum_{(d)|(m), (2) \nmid (d)} N_{K/\mathbb{Q}}(d) \right) = 8 \left(1 + 2 + \frac{15}{8} \sum_{i=0}^{N-1} 4^{2(N-i)+1} \right) \cdot \left(\sum_{(d)|(m), (2) \nmid (d)} N_{K/\mathbb{Q}}(d) \right).$$

Allowing $\left(\sum_{(d)|(m), (2) \nmid (d)} N_{K/\mathbb{Q}}(d) \right) = \mathfrak{S}$, this gives:

$$\begin{aligned} 8 \left(1 + 2 + \frac{15}{8} \sum_{i=0}^{N-1} 4^{2(N-i)+1} \right) \mathfrak{S} &= 8\mathfrak{S} + 16\mathfrak{S} + 15\mathfrak{S} \sum_{i=0}^{N-1} 4^{2(N-i)+1} \\ &= 8\mathfrak{S} + 4 \sum_{(d)|m, \text{ord}_{(2)}(d)=1} N_{K/\mathbb{Q}}(d) + 3(4+1)\mathfrak{S} \sum_{i=0}^{N-1} 4^{2(N-i)+1} \\ &= 8\mathfrak{S} + 4 \sum_{(d)|m, \text{ord}_{(2)}(d)=1} N_{K/\mathbb{Q}}(d) + 3\mathfrak{S} \cdot 4 \sum_{i=2}^{2N+1} 4^i \\ &= 8\mathfrak{S} + 4 \sum_{(d)|m, \text{ord}_{(2)}(d)=1} N_{K/\mathbb{Q}}(d) + 12 \sum_{(d)|m, \text{ord}_{(2)}(d) \geq 2} N_{K/\mathbb{Q}}(d). \end{aligned}$$

The result then follows. The final case behaves similarly and the proof of both Götzky's formula and the general claim that the sum of four squares is universal over $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$ is complete. \square

5. PROOF OF THEOREM 3

We begin by providing a more detailed statement of Theorem 3 than that which appeared in our introduction:

Theorem 3. *Let $K = \mathbb{Q}(\sqrt{D})$ for $D > 0$ a square-free integer and let $m \in \mathcal{O}_K^+$ be locally represented by the sum of four squares Q .*

(a) *If $D \equiv 5 \pmod{8}$, $D \neq 5$. Then (2) is inert and*

$$a_E(m) = \begin{cases} \frac{16D}{5} \left(\sum_{a=1}^{(D-1)/2} \chi_D(a)a(a-D) \right)^{-1} \cdot \left(\sum_{(0) \neq (d)|(m)} N_{K/\mathbb{Q}}(d) \right), & \text{ord}_{(2)}(m) = 0 \\ \frac{16D}{5} \left(\sum_{a=1}^{(D-1)/2} \chi_D(a)a(a-D) \right)^{-1} \cdot 3 \cdot \left(1 + 10 \sum_{i=0}^{N-1} 4^{2(N-i)+1} \right) \cdot \left(\sum_{(d)|(m), (2) \nmid (d)} N_{K/\mathbb{Q}}(d) \right), & \text{ord}_{(2)}(m) = 2N+1, N \geq 0 \\ \frac{16D}{5} \left(\sum_{a=1}^{(D-1)/2} \chi_D(a)a(a-D) \right)^{-1} \cdot 3 \cdot \left(9 + 10 \sum_{i=0}^{N-2} 4^{2(N-i)+1} \right) \cdot \left(\sum_{(d)|(m), (2) \nmid (d)} N_{K/\mathbb{Q}}(d) \right), & \text{ord}_{(2)}(m) = 2N, N \geq 1. \end{cases}$$

Moreover,

$$\frac{192}{5D^{3/2}} \left(\sum_{(d)|(m), (2) \nmid (d)} N_{K/\mathbb{Q}}(d) \right) \leq a_E(m) \leq \frac{8}{5} \begin{cases} \left(\sum_{(d)|(m)} N_{K/\mathbb{Q}}(d) \right), & \text{ord}_{(2)}(m) = 0 \\ 4^6 \left(\sum_{(d)|(m)} N_{K/\mathbb{Q}}(d) \right), & \text{ord}_{(2)}(m) = 2N+1, N \geq 0 \\ 4^5 \left(\sum_{(d)|(m)} N_{K/\mathbb{Q}}(d) \right), & \text{ord}_{(2)}(m) = 2N, N \geq 1. \end{cases}$$

(b) If $D \equiv 1 \pmod{8}$. Then $(2) = \mathfrak{p}_1 \mathfrak{p}_2$ splits and

$$a_E(m) = \begin{cases} \frac{16D}{3} \left(\sum_{a=1}^{(D-1)/2} \chi_D(a) a(a-D) \right)^{-1} \cdot \left(\sum_{(d)|(m)} N_{K/\mathbb{Q}}(d) \right), & \mathfrak{p}_1 \nmid (m), \mathfrak{p}_2 \nmid (m) \\ 16D \left(\sum_{a=1}^{(D-1)/2} \chi_D(a) a(a-D) \right)^{-1} \cdot \left(\sum_{(d)|(m), 2 \nmid N_{K/\mathbb{Q}}(d)} N_{K/\mathbb{Q}}(d) \right), & (2) \nmid (m), N_{K/\mathbb{Q}}(m) \text{ even} \\ 48D \left(\sum_{a=1}^{(D-1)/2} \chi_D(a) a(a-D) \right)^{-1} \cdot \left(\sum_{(d)|(m), 2 \nmid N_{K/\mathbb{Q}}(d)} N_{K/\mathbb{Q}}(d) \right), & (2)|(m). \end{cases}$$

Moreover,

$$\frac{64}{D^{3/2}} \left(\sum_{(d)|(m), (2) \nmid (d)} N_{K/\mathbb{Q}}(d) \right) \leq a_E(m) \leq \frac{8}{3} \left(\sum_{(d)|(m)} N_{K/\mathbb{Q}}(d) \right).$$

(c) If $D \equiv 2, 3 \pmod{4}$. Then $(2) = \mathfrak{p}_2^2$ ramifies and

$$a_E(m) = \begin{cases} 64D \left(\sum_{a=1}^{4D} \chi_{4D}(a) a(a-4D) \right)^{-1} \left(\sum_{(d)|(m), \mathfrak{p}_2 \nmid (d)} N_{K/\mathbb{Q}}(d) \right) \cdot 2^{\text{ord}_{\mathfrak{p}_2}(m)}, & \text{ord}_{\mathfrak{p}_2}(m) = 0, 2 \\ 384D \cdot (2^{2N-1} - 1) \left(\sum_{a=1}^{4D} \chi_{4D}(a) a(a-4D) \right)^{-1} \left(\sum_{(d)|(m), \mathfrak{p}_2 \nmid (d)} N_{K/\mathbb{Q}}(d) \right), & \text{ord}_{\mathfrak{p}_2}(m) = 2N+1, N \geq 1 \\ 384D \cdot (2^{2N-2} - 1) \left(\sum_{a=1}^{4D} \chi_{4D}(a) a(a-4D) \right)^{-1} \left(\sum_{(d)|(m), \mathfrak{p}_2 \nmid (d)} N_{K/\mathbb{Q}}(d) \right), & \text{ord}_{\mathfrak{p}_2}(m) = 2N, N \geq 2. \end{cases}$$

Moreover,

$$\frac{12}{D^{3/2}} \left(\sum_{(d)|(m), (2) \nmid (d)} N_{K/\mathbb{Q}}(d) \right) \leq a_E(m) \leq \begin{cases} 16 \left(\sum_{(d)|(m)} N_{K/\mathbb{Q}}(d) \right), & \text{ord}_{\mathfrak{p}_2}(m) = 0 \\ 96 \left(\sum_{(d)|(m)} N_{K/\mathbb{Q}}(d) \right), & \text{else.} \end{cases}$$

We provide merely a detailed sketch of the proof to avoid repetition. This section is organized first by showing the exact values for $a_E(m)$. Then we provide justification for the lower bounds, and last the upper bounds.

5.1. Exact Values. Let Q be the sum of four squares over K a real quadratic number field of discriminant Δ . Let $m \in \mathcal{O}_K^+$ be locally represented by Q . By the previous sections, we have:

$$\begin{aligned} a_E(m) &= \left(\prod_{\mathfrak{p}|\infty} \beta_{\mathfrak{p}}(m) \right) \left(\prod_{\mathfrak{p} \nmid \mathcal{N}_Q m} \beta_{\mathfrak{p}}(m) \right) \left(\prod_{\mathfrak{p}|\mathcal{N}_Q} \beta_{\mathfrak{p}}(m) \right) \left(\prod_{\mathfrak{p}|m, \mathfrak{p} \nmid \mathcal{N}_Q} \beta_{\mathfrak{p}}(m) \right) \\ &= \left(\frac{\pi^4 N_{K/\mathbb{Q}}(m)}{\Delta^{3/2}} \right) \left(\frac{6\Delta^{5/2}}{\pi^4} \left(\sum_{a=1}^{\Delta} \chi_{\Delta}(a) a(a-\Delta) \right)^{-1} \right) \left(\prod_{\mathfrak{p}|\mathcal{N}_Q} \frac{\beta_{\mathfrak{p}}(m) N_{K/\mathbb{Q}}(\mathfrak{p})^2}{N_{K/\mathbb{Q}}(\mathfrak{p})^2 - 1} \right) \left(\prod_{\mathfrak{p}|(m), \mathfrak{p} \nmid \mathcal{N}_Q} \sum_{i=0}^{\text{ord}_{\mathfrak{p}}(m)} N_{K/\mathbb{Q}}(\mathfrak{p})^{-i} \right). \end{aligned}$$

Suppose first that $K = \mathbb{Q}(\sqrt{D})$ with $0 < D \equiv 5 \pmod{8}$ with D squarefree. We immediately have

$$a_E(m) = \frac{32D}{5} \left(\sum_{a=1}^D \chi_D(a) a(a-D) \right)^{-1} N_{K/\mathbb{Q}}(m) \beta_{(2)}(m) \left(\prod_{\mathfrak{p}|(m), \mathfrak{p} \nmid (2)} \sum_{i=0}^{\text{ord}_{\mathfrak{p}}(m)} N_{K/\mathbb{Q}}(\mathfrak{p})^{-i} \right).$$

Applying Lemma 1 and noting $\chi_D(-a) = \chi_D(a)$ for all a give the exact value of $a_E(m)$ in part (a).

Similarly, if $K = \mathbb{Q}(\sqrt{D})$ with $0 < D \equiv 1 \pmod{8}$ with D squarefree we begin with

$$a_E(m) = \frac{32D}{3} \left(\sum_{a=1}^D \chi_D(a) a(a-D) \right)^{-1} N_{K/\mathbb{Q}}(m) \left(\prod_{\mathfrak{p} \mid (2)} \beta_{\mathfrak{p}}(m) \right) \left(\prod_{\mathfrak{p} \mid (m), \mathfrak{p} \nmid (2)} \sum_{i=0}^{\text{ord}_{\mathfrak{p}}(m)} N_{K/\mathbb{Q}}(\mathfrak{p})^{-i} \right).$$

We now apply Lemma 2, and note that $\chi_D(-a) = \chi_D(a)$ for all a to recover the exact value of $a_E(m)$ provided in part (b).

Last when $K = \mathbb{Q}(\sqrt{D})$ with $0 < D \equiv 2, 3 \pmod{4}$ with D squarefree we have

$$a_E(m) = 32D \left(\sum_{a=1}^{4D} \chi_{4D}(a) a(a-4D) \right)^{-1} N_{K/\mathbb{Q}}(m) \left(\prod_{\mathfrak{p} \mid (2)} \beta_{\mathfrak{p}}(m) \right) \left(\prod_{\mathfrak{p} \mid (m), \mathfrak{p} \nmid (2)} \sum_{i=0}^{\text{ord}_{\mathfrak{p}}(m)} N_{K/\mathbb{Q}}(\mathfrak{p})^{-i} \right).$$

Applying Lemma 3, we recover the exact value of $a_E(m)$ provided in part (c).

5.2. Lower Bounds. We trivially have

$$\zeta(2)^{-1} \leq L_{\mathbb{Q}}(2, \chi_{\Delta})^{-1} \leq \frac{\zeta(2)}{\zeta(4)}$$

or

$$\frac{36}{\pi^4} \leq L_K(2, \chi_{\Delta}) \leq \frac{90}{\pi^4}.$$

Rewriting $a_E(m)$ as

$$a_E(m) = \left(\frac{\pi^4 N_{K/\mathbb{Q}}(m)}{\Delta^{3/2}} \right) L_K(2, \chi_{\Delta})^{-1} \left(\prod_{\mathfrak{p} \mid (2)} \frac{\beta_{\mathfrak{p}}(m) N_{K/\mathbb{Q}}(\mathfrak{p})^2}{N_{K/\mathbb{Q}}(\mathfrak{p})^2 - 1} \right) \left(\prod_{\mathfrak{p} \mid (m), \mathfrak{p} \nmid (2)} \sum_{i=0}^{\text{ord}_{\mathfrak{p}}(m)} N_{K/\mathbb{Q}}(\mathfrak{p})^{-i} \right)$$

this gives

$$a_E(m) \geq \frac{36}{\Delta^{3/2}} N_{K/\mathbb{Q}}(m) \left(\prod_{\mathfrak{p} \mid (2)} \frac{\beta_{\mathfrak{p}}(m) N_{K/\mathbb{Q}}(\mathfrak{p})^2}{N_{K/\mathbb{Q}}(\mathfrak{p})^2 - 1} \right) \left(\prod_{\mathfrak{p} \mid (m), \mathfrak{p} \nmid (2)} \sum_{i=0}^{\text{ord}_{\mathfrak{p}}(m)} N_{K/\mathbb{Q}}(\mathfrak{p})^{-i} \right).$$

When $K = \mathbb{Q}(\sqrt{D})$ for $0 < D \equiv 5 \pmod{8}$ D squarefree this implies

$$a_E(m) \geq \frac{192}{5D^{3/2}} N_{K/\mathbb{Q}}(m) \beta_{(2)}(m) \left(\prod_{\mathfrak{p} \mid (m), \mathfrak{p} \nmid (2)} \sum_{i=0}^{\text{ord}_{\mathfrak{p}}(m)} N_{K/\mathbb{Q}}(\mathfrak{p})^{-i} \right).$$

The lower bound from part (a) of Theorem 3 follows from Lemma 1.

Deriving the lower bounds for the remaining cases behaves similarly; note for $D \equiv 2, 3 \pmod{4}$, one must use the substitution $\Delta = 4D$.

5.3. Upper Bounds. Proceeding now towards the upper bounds on $a_E(m)$, we begin with a lemma:

Lemma 5. *Let $0 < D \equiv 1 \pmod{4}$ with D squarefree and let χ_D denote the quadratic Dirichlet character of conductor D . Then*

$$\left(\sum_{a=1}^{(D-1)/2} \chi_D(a) a(a-D) \right) \equiv 0 \pmod{2}, \text{ and if } D > 5, \left(\sum_{a=1}^{(D-1)/2} \chi_D(a) a(a-D) \right) \equiv 0 \pmod{D}.$$

Similarly, for $D \equiv 2, 3 \pmod{4}$ with D squarefree and let χ_D denote the quadratic Dirichlet character of conductor $4D$. We have

$$\left(\sum_{a=1}^{2D} \chi_{4D}(a) a(a-4D) \right) \equiv 0 \pmod{2} \text{ and } \left(\sum_{a=1}^{4D} \chi_{4D}(a) a(a-4D) \right) \equiv 0 \pmod{D}.$$

Proof. That the value is even is obvious. That it is divisible by D is not. First consider $D \equiv 1 \pmod{4}$ prime. Suppose $1 \neq q \in (\mathbb{Z}/D\mathbb{Z})$ is a primitive root. Then

$$\begin{aligned}\chi_D(q)q^2 \sum_{a=0}^{D-1} \chi_D(a)a^2 &\equiv \sum_{a=0}^{D-1} \chi_D(qa)(qa)^2 \pmod{D} \\ &\equiv \sum_{b=0}^{D-1} \chi_D(b)b^2 \pmod{D}\end{aligned}$$

and

$$(\chi_D(q)q^2 - 1) \sum_{a=0}^{D-1} \chi_D(a)a^2 \equiv 0 \pmod{D}.$$

If $(\chi_D(q)q^2 - 1) \equiv 0 \pmod{D}$ then $\chi_D(q)q^2 \equiv 1 \pmod{D}$, and as $D \neq 5$ then we have a contradiction. So since $(\chi_D(q)q^2 - 1) \not\equiv 0 \pmod{D}$, $2 \sum_{a=0}^{(D-1)/2} \chi_D(a)a^2 \equiv \sum_{a=0}^{D-1} \chi_D(a)a^2 \equiv 0 \pmod{D}$ and the claim holds for D prime.

The remaining cases are quite similar. Instead of setting q to be a primitive root, one selects $1 \neq q \in (\mathbb{Z}/D\mathbb{Z})^\times$ such that for all odd $p|D$, $\chi_D(q)q^2 \not\equiv 1 \pmod{p}$. For any prime $p \neq 5$ that such a q can be chosen is obvious. As for $p = 5$, note that if $\chi_D(q)q^2 \equiv 1 \pmod{5}$ for all $q \in (\mathbb{Z}/D\mathbb{Z})^\times$ then $\chi_D(q) \equiv q^2 \pmod{5}$, which implies that $\chi_D(q) = \chi_5(q)$. As $D \neq 5$, this is impossible. \square

Now for a justification of the upper bounds of Theorem 3. First let $D \equiv 5 \pmod{8}$. The constant $\frac{8}{5}$ in the upper bound of part (a) of Theorem 3 comes immediately from Lemma 5, noting that $\left(\sum_{a=1}^{(D-1)/2} \chi_D(a)a(a-D) \right)^{-1} \leq \frac{1}{2D}$. Suppose now that $\text{ord}_{(2)}(m) = 2N + 1$, $N \geq 0$. Then

$$\begin{aligned}3 \cdot \left(1 + 10 \sum_{i=0}^{N-1} 4^{2(N-i+1)+1} \right) &= 3 + 10(4^{2N+4} - 4^{N+4}) \\ &= 4^{2N+1} (4^3(10 - 1/4^N + 3/4^{2N-2})) \\ &\leq 4^{2N+1}(4^6).\end{aligned}$$

As $4^{2N+1} = N_{K/\mathbb{Q}}(2)^{\text{ord}_{(2)}(m)}$, the result follows. A similar argument finishes the remaining case of $\text{ord}_{(2)}(m) = 2N$ and the proof of part (a) of Theorem 3 is complete.

Suppose next that $D \equiv 1 \pmod{8}$. Again, by Lemma 5 the upper bound when $N_{K/\mathbb{Q}}(m)$ is odd is immediate. Suppose now that $(2) \nmid (m)$, $N_{K/\mathbb{Q}}(m)$ is even. Then

$$\begin{aligned}a_E(m) &= 16D \left(\sum_{a=1}^{(D-1)/2} \chi_D(a)a(a-D) \right)^{-1} \cdot \left(\sum_{(d)|(m), 2 \nmid N_{K/\mathbb{Q}}(d)} N_{K/\mathbb{Q}}(d) \right) \\ &\leq 8 \left(\sum_{(d)|(m), 2 \nmid N_{K/\mathbb{Q}}(d)} N_{K/\mathbb{Q}}(d) \right) \\ &\leq \frac{8}{3} \left(\left(\sum_{(d)|(m), 2 \nmid N_{K/\mathbb{Q}}(d)} N_{K/\mathbb{Q}}(d) \right) + 2 \left(\sum_{(d)|(m), 2 \nmid N_{K/\mathbb{Q}}(d)} N_{K/\mathbb{Q}}(d) \right) \right) \\ &\leq \frac{8}{3} \left(\sum_{(d)|(m)} N_{K/\mathbb{Q}}(d) \right).\end{aligned}$$

The upper bound from part (b) therefore holds. At this point, the proof of Corollary 2 follows from the fact that $r_Q(1) = 8$ for all real quadratic number fields while for $D \equiv 5 \pmod{8}$ $a_E(1) \leq \frac{8}{5}$ and for $D \equiv 1 \pmod{8}$ $a_E(1) \leq \frac{8}{3}$.

Finally, we have $D \equiv 2, 3 \pmod{4}$. Applying Lemma 5 to part (c) of Theorem 3 we have:

$$a_E(m) \leq \begin{cases} 16 \left(\sum_{(d)|(m), (2) \nmid (d)} N_{K/\mathbb{Q}}(d) \right) \cdot 2^{\text{ord}_{p_2}(m)}, & \text{ord}_{p_2}(m) = 0, 2 \\ 96 \cdot (2^{2N-1} - 1) \left(\sum_{(d)|(m), p_2 \nmid (d)} N_{K/\mathbb{Q}}(d) \right), & \text{ord}_{p_2}(m) = 2N + 1, N \geq 1 \\ 96 \cdot (2^{2N-2} - 1) \left(\sum_{(d)|(m), p_2 \nmid (d)} N_{K/\mathbb{Q}}(d) \right), & \text{ord}_{p_2}(m) = 2N, N \geq 2 \end{cases}$$

and the result follows immediately.

6. PROOF OF THEOREM 2

We recall first the statement of Theorem 2:

Theorem 2 Let $K = \mathbb{Q}(\sqrt{2})$ and suppose m is a totally positive integer which is locally represented by the sum of four squares Q . The number $r_Q(m)$ of ways m can be expressed as the sum of four O_K -squares is

$$r_Q(m) = 8 \sum_{0 \neq (d)|m} N_{K/\mathbb{Q}}(d) - 6 \sum_{(2)|(d)|m} N_{K/\mathbb{Q}}(d) + 4 \sum_{(4)|(d)|m} N_{K/\mathbb{Q}}(d).$$

Proof. Using part (c) of Theorem 3, we begin with

$$a_E(m) = \begin{cases} 8 \left(\sum_{(0) \neq (d)|(m), p_2 \nmid (d)} N_{K/\mathbb{Q}}(d) \right) \cdot 2^{\text{ord}_{p_2}(m)}, & \text{ord}_{p_2}(m) = 0, 2 \\ 48 \cdot (2^{2N-1} - 1) \left(\sum_{(0) \neq (d)|(m), p_2 \nmid (d)} N_{K/\mathbb{Q}}(d) \right), & \text{ord}_{p_2}(m) = 2N + 1, N \geq 1 \\ 48 \cdot (2^{2N-2} - 1) \left(\sum_{(0) \neq (d)|(m), p_2 \nmid (d)} N_{K/\mathbb{Q}}(d) \right), & \text{ord}_{p_2}(m) = 2N, N \geq 2. \end{cases}$$

Using Magma [22], one confirms that the space of Hilbert modular cusp forms of parallel weight 2, level (4) and trivial character over $\mathbb{Q}(\sqrt{2})$ is zero dimensional. Therefore, for any locally represented $m \in O_K^+$, $r_Q(m) = a_E(m)$. Thus, just as in the proof of Theorem 2, we are reduced to showing that the exact formula for $a_E(m)$ can be simplified to the formula for $r_Q(m)$ given above. Again, we start by rewriting the sum. Let d be a divisor of m :

$$8 \sum_{0 \neq (d)|m} N_{K/\mathbb{Q}}(d) - 6 \sum_{(2)|(d)|m} N_{K/\mathbb{Q}}(d) + 4 \sum_{(4)|(d)|m} N_{K/\mathbb{Q}}(d) = 8 \left(\sum_{\text{ord}_{p_2}(d)=0,1} N_{K/\mathbb{Q}}(d) \right) + 2 \left(\sum_{\text{ord}_{p_2}(d)=2,3} N_{K/\mathbb{Q}}(d) \right) + 6 \left(\sum_{\text{ord}_{p_2}(d) \geq 4} N_{K/\mathbb{Q}}(d) \right).$$

We see immediately, then, that when $\text{ord}_{p_2}(m) = 0, 2$, the formula holds.

Suppose next that $\text{ord}_{p_2}(m) = 2N + 1$, $N \geq 1$. As in the proof of Theorem 2, set $\mathfrak{S} = \left(\sum_{\text{ord}_{p_2}(d)=0} N_{K/\mathbb{Q}}(d) \right)$.

Then

$$\begin{aligned} 8 \sum_{0 \neq (d)|m} N_{K/\mathbb{Q}}(d) - 6 \sum_{(2)|(d)|m} N_{K/\mathbb{Q}}(d) + 4 \sum_{(4)|(d)|m} N_{K/\mathbb{Q}}(d) &= 8(1 + 2)\mathfrak{S} + 2(4 + 8)\mathfrak{S} + 6(2^4 + 2^5 + \dots + 2^{2N+1})\mathfrak{S} \\ &= 48(1)\mathfrak{S} + 48(2 + 2^2 + \dots + 2^{2N-2})\mathfrak{S} \\ &= 48(2^{2N-1} - 1)\mathfrak{S} = a_E(m). \end{aligned}$$

So the formula holds for $\text{ord}_{p_2}(m)$ odd. The remaining case is left to the reader. \square

7. ADDITIONAL EXAMPLES

Example 1. Let $K = \mathbb{Q}(\sqrt{3})$. Part (c) of Theorem 3 gives

$$a_E(m) = \begin{cases} 4 \left(\sum_{(0) \neq (d)|(m), (2) \nmid (d)} N_{K/\mathbb{Q}}(d) \right), & \mathfrak{p}_2 \nmid (m) \\ \frac{87}{4} \cdot (2^{\text{ord}_{\mathfrak{p}_2}(m)-2} - 1) \left(\sum_{(0) \neq (d)|(m), \mathfrak{p}_2 \nmid (d)} N_{K/\mathbb{Q}}(d) \right), & \text{ord}_{\mathfrak{p}_2}(m) \geq 1. \end{cases}$$

Magma [22] computations show that the dimension of the space of Hilbert modular cusp forms of parallel weight 2 and level (4) over $\mathbb{Q}(\sqrt{3})$ is one, with that form being a normalized newform $f(z)$. As $a_E(1) = 4 = \frac{1}{2}r_Q(1)$, we see that we have

$$\Theta_Q(z) = E(z) + 4f(z).$$

Example 2. Let $K = \mathbb{Q}(\sqrt{13})$. Part (a) of Theorem 3 gives

$$a_E(m) = \begin{cases} \frac{8}{5} \cdot \left(\sum_{(0) \neq (d)|(m)} N_{K/\mathbb{Q}}(d) \right), & (2) \nmid (m) \\ \frac{24}{5} \cdot \left(1 + 10 \sum_{i=0}^{N-1} 4^{2(N-i+1)+1} \right) \cdot \left(\sum_{(0) \neq (d)|(m), (2) \nmid (d)} N_{K/\mathbb{Q}}(d) \right), & \text{ord}_{(2)}(m) = 2N+1, N \geq 0 \\ \frac{24}{5} \cdot \left(9 + 10 \sum_{i=0}^{N-2} 4^{2(N-i+1)} \right) \cdot \left(\sum_{(0) \neq (d)|(m), (2) \nmid (d)} N_{K/\mathbb{Q}}(d) \right), & \text{ord}_{(2)}(m) = 2N, N \geq 1. \end{cases}$$

Note that in particular, by considering $m = 1$ this shows that the upper bound for $a_E(m)$ given in Theorem 3 is sharp. Using Magma [22] we find that while the space of Hilbert modular cusp forms of parallel weight 2 and level (4) over $\mathbb{Q}(\sqrt{13})$ is two dimensional, there are no newforms. We note that the dimensions of the spaces of cusp forms of levels (2) and (1) are (respectively) one and zero, and we call f the level (2) normalized eigenform and $f_{(2),(2)}$ the resulting oldform of level (4). Thus

$$\Theta_Q(m) = E(z) + \frac{32}{5}f(z) + \frac{128}{5}f_{(2),(2)}(z).$$

Example 3. Let $K = \mathbb{Q}(\sqrt{17})$. Part (b) of Theorem 3 gives

$$a_E(m) = \begin{cases} \frac{4}{3} \left(\sum_{(0) \neq (d)|(m)} N_{K/\mathbb{Q}}(d) \right), & \mathfrak{p}_1 \nmid (m), \mathfrak{p}_2 \nmid (m) \\ 4 \left(\sum_{(0) \neq (d)|(m), 2 \nmid N_{K/\mathbb{Q}}(d)} N_{K/\mathbb{Q}}(d) \right), & (2) \nmid (m), N_{K/\mathbb{Q}}(m) \text{ even} \\ 12 \left(\sum_{(0) \neq (d)|(m), 2 \nmid N_{K/\mathbb{Q}}(d)} N_{K/\mathbb{Q}}(d) \right), & (2)|(m). \end{cases}$$

The corresponding space of Hilbert modular cusp forms is 5-dimensional. There is one normalized newform at (4) which we call $f_4(z)$. There is additionally one normalized newform of (2), f_2 , from which we obtain the remainder of the space by lifts (with the resulting forms dubbed $f_{2,2}, f_{2,\mathfrak{p}_1}, f_{2,\mathfrak{p}_2}$). We collect $r_Q(m)$ and $a_E(m)$ values to solve the necessary system of linear equations to find that

$$\Theta_Q(z) = E(z) + \frac{16}{3}(f_{2,2}(z) + f_4(z)) + \frac{4}{3}f_2(z) - \frac{8}{3}(f_{2,\mathfrak{p}_1} + f_{2,\mathfrak{p}_2}).$$

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